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TWO APPLICATIONS OF APPROXIMATE DIFFEFENTIATION FORMULAE: AN EXTREMUM PROBLEM FOR MULTIPLY MONOTONE FUNCTIONS AND THE DIFFERENTIATION OF ASYMPTOTIC EXPANSIONS

I. J. Schoenberg

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Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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TWO APPLICATIONS OF APPROXIMATE DIFFERENTIATION FORMULAE: AN EXTREMUM PROBLEM FOR MULTIPLY MONOTONE FUNCTIONS AND THE DIFFERENTIATION OF ASYMPTOTIC EXPANSIONS

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ABSTRACT

This report presents two further applications of very elementary formulae of approximate differentiation. The first is a new derivation in a somewhat sharper form of the following theorem of V. M. Olovyanišnikov:

Let $N_n (n \geqq 2)$ be the class of functions $g(x)$ such that $g(x)$, $g'(x),\ldots,g^{(n)}(x)$ are $\geqq 0$, bounded, and non-decreasing on the half-line $-\infty < x \leqq 0$. A special element of N_n is

$$g_*(x) = 0 \quad \text{if} \quad -\infty < x < -1, \quad g_*(x) = (1+x)^n \quad \text{if} \quad -1 \leqq x \leqq 0 \quad .$$

If $g(x) \in N_n$ is such that

$$g(0) < g_*(0) = 1, \quad g^{(n)}(0) < g_*^{(n)}(0) = n!$$

then

$$g^{(\nu)}(0) \leqq g_*^{(\nu)}(0) \quad \text{for} \quad \nu = 1,\ldots,n-1 \quad . \tag{1}$$

Moreover, if we have equality in (1) for some value of ν, then we have there equality for all ν, and this happens only if $g(x) = g_*(x)$ in $(-\infty,0]$.

The second application gives sufficient conditions for the differentiability of asymptotic expansion (Theorem 4).

AMS(MOS) Subject Classifications: 41A17, 65D25.

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SIGNIFICANCE AND EXPLANATION

This report presents two further applications of very elementary formulae of approximate differentiation. The first is a new derivation in a somewhat sharper form of a theorem of V. M. Olovyanišnikov on multiply monotone functions. The second applications gives sufficient conditions for the differentiability of asymptotic expansions.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

TWO APPLICATIONS OF APPROXIMATE DIFFERENTIATION FORMULAE:

AN EXTREMUM PROBLEM FOR MULTIPLY MONOTONE FUNCTIONS

AND THE DIFFERENTIATION OF ASYMPTOTIC EXPANSIONS

I. J. Schoenberg

1. Introduction. Approximate differentiation formulae are admirably suited for the solution of certain extremum problems of the Landau-Kolmogorov type (see [5] and [1]). Here we deal with two unrelated further applications.

In Part I we establish Theorem 1 below due to V. M. Olovyanišnikov [4] (see R. P. Boas' review in Math. Reviews 13, p. 17. I owe to Boas the reference to this paper). We restate Theorem 1 as Theorem 2 in terms of multiply monotone functions on the positive real axis, at the same time showing the unicity of the extremizing functions (§3). In §4 we use a result from R. E. Williamson's paper [7] to derive a second proof of Olovyanisnikov's theorem.

Part II deals with our second topic. We derive sufficient conditions for the differentiation of asymptotic expansions.

I. Multiply monotone functions.

2. Olovyanišnikov's theorem. Let N_n $(n \geq 2)$ denote the class of functions $g(x)$ defined on $(-\infty,0]$ such that

$$(2.1) \qquad g(x),\ g'(x),\ldots,g^{(n)}(x) \ \underline{\text{are}}\ \geq 0,\ \underline{\text{bounded,}}$$
$$\underline{\text{and non-decreasing on}}\ (-\infty,0]\ .$$

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Here the first n functions are assumed absolutely continuous, while $g^{(n)}(x)$ may not exist for a countable number of values of x. A special element of N_n is the function

$$(2.2) \qquad g_*(x) = \begin{cases} 0 & \text{if } -\infty < x < -1 , \\ (1+x)^n & \text{if } -1 \leqq x \leqq 0 . \end{cases}$$

An extremizing property of this special function is stated in the following

Theorem 1 (Olovyanišnikov). *If*

$$(2.3) \qquad g(x) \in N_n$$

and

$$(2.4) \qquad g(0) \leqq g_*(0) = 1, \; g^{(n)}(0) \leqq g_*^{(n)}(0) = n! ,$$

then

$$(2.5) \qquad g^{(\nu)}(0) \leqq g_*^{(\nu)}(0) \quad \text{for} \quad \nu = 1,2,\ldots,n-1 .$$

To this we add now the following new unicity property of the extremizing function $g_*(x)$:

Unicity theorem. *If we have equality in* (2.5) *for some value of* ν, *then the equality holds for all* ν, *and this happens only if*

$$(2.6) \qquad g(x) = g_*(x) \quad \text{in} \quad [-\infty, 0] .$$

The class N_n seems to be the right setting for these remarkable results. However, to carry out our new approach we prefer to pass from x to $-x$, writing $f(x) = g(-x)$, and to state the following definitions.

Let $M_n (n \geq 2)$ denote the class of functions $f(x)$, $0 \leq x < \infty$, such that

$$(2.7) \qquad (-1)^{\nu} f^{(\nu)}(x) \ (\nu = 0,\ldots,n) \ \underline{\text{are}} \ \geq 0, \ \underline{\text{bounded,}} \ \underline{\text{and non-increasing on}} \ [0,\infty) \ .$$

This is the class of n-times monotone functions on $[0,\infty)$. A special element of M_n is the function

$$(2.8) \qquad f_*(x) = (1-x)_+^n \ , \quad (x \leq 0) \ ,$$

where $t_+ = \max(t,0)$.

We may now restate all previous results as follows.

Theorem 2. If

$$(2.9) \qquad f(x) \in M_n$$

and

$$(2.10) \qquad f(0) \leq f_*(0) = 1, \ (-1)^n f^{(n)}(0) \leq (-1)^n f_*^{(n)}(0) = n! \ ,$$

then

$$(2.11) \qquad (-1)^{\nu} f^{(\nu)}(0) \leq (-1)^{\nu} f_*^{(\nu)}(0) \ \underline{\text{for}} \ \nu = 1,2,\ldots,n-1 \ .$$

Moreover, if we have equality in (2.11) *for some value of* ν, *then equality holds for all* ν, *and this happens only if*

$$f(x) = f_*(x) \quad \text{in} \quad [0,\infty) \quad . \tag{2.12}$$

3. A proof of Theorem 2. We use a formula of approximate differentiation that expresses $f^{(\nu)}(0)$ as an appropriate linear combination of the n data

$$f(0),\ f(1),\ f'(1),\ldots,f^{(n-2)}(1) \quad , \tag{3.1}$$

and is such that the formula should be exact for polynomials of degree $n - 1$. Such a formula can easily be obtained from Peano's theorem (see [2, §3.7]), but in this case we may derive it directly as follows.

Taylor's formula gives the identity

$$f(0) = \sum_{0}^{n-1} \frac{(-1)^k}{k!} f^{(k)}(1) + \frac{(-1)^n}{(n-1)!} \int_0^1 x^{n-1} f^{(n)}(x)\,dx \quad . \tag{3.2}$$

Applying this to $f^{(\nu)}(x)$, rather than $f(x)$, and for $n - \nu$ in place of n, we obtain

$$f^{(\nu)}(0) = \sum_{\nu}^{n-1} \frac{(-1)^{k-\nu}}{(k-\nu)!} f^{(k)}(1) + \frac{(-1)^{n-\nu}}{(n-\nu-1)!} \int_0^1 x^{n-\nu-1}\ f^{(n)}(x)\,dx \quad . \tag{3.3}$$

In view of the data (3.1) we eliminate the unwanted term $f^{(n-1)}(1)$ between these relations: Multiplying (3.2) by $-(n-1)(n-2)\cdots(n-\nu)$, and (3.3) by $(-1)^{\nu}$, and adding them, we finally obtain the identity

$$(3.4)\qquad \begin{aligned}(-1)^{\nu} f^{(\nu)}(0) &= \frac{(n-1)!}{(n-\nu-1)!}\left\{f(0) - \sum_{k=0}^{\nu-1} \frac{1}{k!}(-1)^k f^{(k)}(1)\right\} \\ &\quad - \sum_{k=\nu}^{n-2} A_k (-1)^k f^{(k)}(1) + \int_0^1 K(x)(-1)^n f^{(n)}(x)\,dx ,\end{aligned}$$

where

$$(3.5)\qquad A_k = \frac{(n-1)!}{(n-\nu-1)!} - \frac{k!}{(k-\nu)!} > 0 \quad \text{for} \quad k = \nu, \nu+1, \ldots, n-2$$

and

$$(3.6)\qquad K(x) = \frac{1}{(n-\nu-1)!} x^{n-\nu-1}(1-x^{\nu}) > 0 \quad \text{in} \quad (0,1) .$$

Notice that the second sum on the right side of (3.4) is absent if $\nu = n-1$.

All the desired conclusion follow almost immediately from the identity (3.4) in view of the positivity of the A_k and $K(x)$.

(i) If $f(x) = f_*(x) = (1-x)_+^n$, then (3.4) gives

$$(3.7)\qquad \begin{aligned}\frac{n!}{(n-\nu)!} &= (-1)^{\nu} f_*^{(\nu)}(0) = \\ &\frac{(n-1)!}{(n-\nu-1)!}\left\{1 + \sum_{k=0}^{\nu-1} 0\right\} + \sum_{k=\nu}^{n-2} 0 + n!\int_0^1 K(x)\,dx .\end{aligned}$$

On the other hand, if $f(x) \in M_n$ and satisfies (2.10), then (3.4) and (3.7) show that

$$(3.8)\qquad \begin{aligned}(-1)^{\nu} f^{(\nu)}(0) &\leq \frac{(n-1)!}{(n-\nu-1)!}\left\{1 + \sum_{k=0}^{\nu-1} 0\right\} + \sum_{k=\nu}^{n-2} 0 + n!\int_0^1 K(x)\,dx \\ &= (-1)^{\nu} f_*^{(\nu)}(0) ,\end{aligned}$$

establishing (2.11).

(ii) When does the equality sign hold in (3.8)? In view of (3.4) and (2.10) this will be the case if and only if the following conditions are simultaneously satisfied:

$$f(0) = 1,\ f(1) = f'(1) = \ldots = f^{(n-2)}(1) = 0 \quad , \tag{3.9}$$

and

$$(-1)^n f^{(n)}(x) = n! \quad \text{almost everywhere in} \quad (0,1) \quad . \tag{3.10}$$

Now (3.10) shows that $f(x)$ is a polynomial of degree n, while the equations (3.9) show that it must be of the form

$$f(x) = (1-x)^{n-1}(1-ax) \quad .$$

Finally, again (3.10) shows that we must have $a = 1$. Thus the identity (2.12) is established.

4. <u>A second proof of Theorem 2.</u> Here we use a fundamental result from Williamson's paper [7] which we state as

<u>Lemma 1.</u> <u>The class</u> $M_n = \{f(x)\}$ <u>defined by</u> (2.7), <u>is identical with the class of functions</u> $f(x)$ <u>admitting a Stieltjes integral representation</u>

$$f(x) = \int_0^{\infty} (1-xt)_+^n \, d\alpha(t) \quad , \tag{4.1}$$

<u>where</u>

(4.2) $\alpha(t)$ <u>is non-decreasing and bounded on</u> $[0,\infty)$, $\alpha(0) = 0$,

<u>and</u>

$$\mu_n = \int_0^\infty t^n d\alpha(t) < \infty \quad . \tag{4.3}$$

There is a close relationship between the discontinuities of the monotone functions $\alpha(t)$ and $f^{(n)}(x)$: $t = \tau > 0$ is a discontinuity of $\alpha(t)$ if and only if $x = 1/\tau$ is a discontinuity of $f^{(n)}(x)$. However, $f^{(n)}(x)$ is continuous at $x = 0$, hence $f^{(n)}(0) = f^{(n)}(0+)$.

The derivatives $f^{(\nu)}(x)$ are obtained by differentiation of (4.1) with the result that

$$(-1)^\nu f^{(\nu)}(x) = c_{n,\nu} \int_0^\infty (1-xt)_+^{n-\nu} t^\nu d\alpha(t), \; (\nu = 0,\ldots,n) \quad , \tag{4.4}$$

where $c_{n,\nu} = n(n-1)\ldots(n-\nu+1)$, $c_{n,0} = 1$, and in particular

$$(-1)^\nu f^{(\nu)}(0) = c_{n,\nu} \int_0^\infty t^\nu d\alpha(t), \; (\nu = 0,\ldots,n) \quad . \tag{4.5}$$

We obtain the special function $f_*(x) = (1-x)_+^n$ from (4.1) if

$$\alpha(t) = \alpha_*(t) = \begin{cases} 0 & \text{if } 0 \leqq t < 1 \; , \\ 1 & \text{if } t \geqq 1 \; , \end{cases} \tag{4.6}$$

and therefore

$$(-1)^\nu f_*^{(\nu)}(0) = c_{n,\nu} \cdot 1, \quad (\nu = 0,\ldots,n) \quad . \tag{4.7}$$

It follows that in terms of $\alpha(t)$ the assumptions (2.10) are equivalent to the inequalities

$$\mu_0 = \int_0^\infty d\alpha(t) \leqq 1 \quad , \quad \mu_n = \int_0^\infty t^n d\alpha(t) \leqq 1 \quad ; \tag{4.8}$$

to establish Theorem 2 we are to show that (4.8) imply the inequalities

$$\mu_\nu = \int_0^\infty t^\nu d\alpha(t) \leqq 1 \quad \text{for} \quad \nu = 1,2,\dots,n-1 \quad . \tag{4.9}$$

This follows immediately from a known property of the moments μ_ν:

(4.10) <u>The sequence</u> $\mu_\nu(\nu = 0,\dots,n)$ <u>is logarithmically convex</u>,

which means that

(4.11) <u>the sequence</u> $\log \mu_\nu(\nu = 0,\dots,n)$ <u>is convex</u>.

For completeness we sketch a proof of the property (4.11): <u>It amounts to proving the inequalities</u>

$$(\mu_\nu)^2 \leqq \mu_{\nu-1}\mu_{\nu+1} \quad \text{for} \quad \nu = 1,\dots,n-1 \quad . \tag{4.12}$$

It is clearly sufficient to establish (4.12) for the case when $\alpha(t)$ is a non-decreasing step-function with a finite number of discontinuities, in which case

$$\mu_\nu = \sum_{j=1}^m c_j b_j^\nu, \ 0 \leqq b_1 < b_2 < \dots < b_m, \quad \text{all} \quad c_j > 0 \quad .$$

By Cauchy's inequalities we have

$$\mu_\nu^2 = \left(\sum_1^m c_j b_j^\nu\right)^2 = \left(\sum_1^m c_j^{1/2} b_j^{(\nu-1)/2} \cdot c_j^{1/2} b_j^{(\nu+1)/2}\right)^2$$

$$\leqq \left(\sum_1^m c_j b_j^{\nu-1}\right)\left(\sum_1^m c_j b_j^{\nu+1}\right) = \mu_{\nu-1}\mu_{\nu+1}$$

establishing (4.12).

Let us finally establish the last sentence of Theorem 2: <u>We assume that we have the equality sign in</u> (4.9) <u>for some value of</u> ν, <u>and we are to show that</u> $\alpha(t) = \alpha_+(t)$.

From the convexity property (4.11) the assumptions $\mu_0 \leqq 1$ and $\mu_n \leqq 1$, and the equation $\mu_\nu = 1$, for some ν, evidently imply that

$$\mu_\nu = \int_0^\infty t^\nu d\alpha(t) = 1 \quad \text{for} \quad \nu = 0,\ldots,n \quad , \tag{4.13}$$

and we are to conclude that

$$\alpha(t) = \alpha_*(t) = \begin{cases} 0 & \text{if} \quad 0 \leqq t < 1 \quad , \\ 1 & \text{if} \quad t \geqq 1 \quad . \end{cases} \tag{4.14}$$

<u>Proof:</u> We distinguish two cases depending on the parity of n. <u>If</u> n <u>is even</u> we observe that (4.13) imply that

$$\int_0^\infty (t-1)^n d\alpha(t) = \sum_0^n (-1)^\nu \binom{n}{\nu} = 0 \quad .$$

The integrand being positive if $t \neq 1$, this implies (4.14). <u>If</u> n <u>is odd</u> we similarly conclude that

$$\int_0^\infty t(1-t)^{n-1} d\alpha(t) = 0 \quad .$$

Again, the integrand being positive if $t \neq 0$ and $t \neq 1$, we conclude that $(0,1)$ and $(1,\infty)$ are intervals of constancy of $\alpha(t)$. Thus $\alpha(t)$ is a step-function having non-negative jumps A_0 and A_1 at $t = 0$ and $t = 1$, respectively. Now (4.13), for $\nu = 0,1$, show that $A_0 + A_1 = 1$, $A_1 = 1$, hence $A_0 = 0$ and (4.14) is established.

II. Asymptotic Expansions

5. The differentiation of asymptotic estimates. Theorem 3 below was suggested by a theorem of Hardy-Littlewood-Landau in a version due to Widder [6, 223]. This theorem is as follows:

Let $f(x) \in C^2(0,\infty)$ *such that*

$$(A_1) \qquad f(x) \sim \frac{a_1}{x} \quad (x \to \infty) \quad ,$$

$$(A_2) \qquad f''(x) = O(x^{-3})$$

Then

$$(C) \qquad f'(x) \sim -\frac{a_1}{x^2} \quad .$$

Widder proves it elegantly by applying one of Wiener's Tauberian theorems.

This result becomes more elementary if we modify its statement by replacing the assumption (A_1) by the stronger assumption

$$(A_1') \qquad f(x) = \frac{a_1}{x} + \frac{a_2}{x^2} + 0(x^{-3}) \quad .$$

We shall show that (A_1') and (A_2) imply that

$$(C') \qquad f'(x) = -\frac{a_1}{x^2} + 0(x^{-3}) \quad ,$$

which is stronger than (C).

This modified theorem also generalizes to higher derivatives and we may state our

<u>**Theorem 3.**</u> <u>Let</u> $f(x) \in C^n(0,\infty)$, $(n \geq 2)$, <u>and let us assume that</u>

$$(5.1) \qquad f(x) = \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots + \frac{a_n}{x^n} + 0\left(x^{-n-1}\right) \quad ,$$

$$(5.2) \qquad f^{(n)}(x) = 0(x^{-n-1}) \quad .$$

<u>Then</u>

$$(C_1) \qquad f'(x) = -\frac{a_1}{x^2} - \frac{2a_2}{x^3} - \ldots - \frac{(n-1)a_{n-1}}{x^n} + 0(x^{-n-1})$$

$$(C_2) \qquad f''(x) = \frac{2a_1}{x^3} + \ldots + \frac{(n-2)(n-1)a_{n-2}}{x^n} + 0(x^{-n-1})$$

$$\vdots$$

$$(C_{n-1}) \qquad f^{(n-1)}(x) = (-1)^{n-1}\frac{(n-1)!a_1}{x^n} + 0(x^{-n-1}) \quad .$$

In words: <u>The assumption</u> (5.2) <u>insures that we can differentiate formally the asymptotic estimate</u> (5.1) $n-1$ <u>times.</u>

__Proof:__ Writing as usual $(a)_n = a(a+1)\cdots(a+n-1)$, let us establish the estimate

$$(C_k) \qquad f^{(k)}(x) = (-1)^k \frac{(1)_k a_1}{x^{k+1}} + \dots + (-1)^k \frac{(n-k)_k a_{n-k}}{x^n} + O(x^{-n-1}) \quad ,$$

for $k = 1,\dots,n-1$. To obtain it we need a formula for approximate differentiation of the form

$$(5.3) \qquad F^{(k)}(0) = \sum_0^{n-1} A_\nu(F(\nu) + R(F) \quad .$$

Here the A_ν are appropriate numerical constants and $R(F)$ is a linear functional which is to vanish whenever $F \in \pi_{n-1}$. We do not need the values of the A_ν, but only wish to clarify their existence: If

$$(5.4) \qquad P(x) = \sum_0^{n-1} P(\nu)\ell_\nu(x), \text{ where } P \in \pi_{n-1} \quad ,$$

is Lagrange's interpolation formula, then also

$$P^{(k)}(x) = \sum_0^{n-1} P(\nu)\ell_\nu^{(k)}(x) \quad ,$$

and in particular

$$P^{(k)}(0) = \sum_0^{n-1} P(\nu)\ell_\nu^{(k)}(0), \text{ whence } A_\nu = \ell_\nu^{(k)}(0) \quad .$$

By Peano's theorem (see [2, §3.7]) we may rewrite (5.3) as

$$(5.5) \qquad F^{(k)}(0) = \sum_0^{n-1} A_\nu F(\nu) + \int_0^{n-1} K(t)F^{(n)}(t)\,dt \quad .$$

Here $K(x)$ is a kernel that can be determined by Peano's theorem, but whose explicit form we do not need. Applying (5.5) to the function $F(t) = f(x+t)$ we obtain the identity

$$f^{(k)}(x) = \sum_{0}^{n-1} A_\nu f(x+\nu) + \int_{0}^{n-1} K(t) f^{(n)}(x+t)\,dt, \quad (x > 0) \quad . \tag{5.6}$$

To simplify notations we use the symbols

$$\Delta(x;f) = \sum_{0}^{n-1} A_\nu f(x+\nu) \quad \text{and} \quad \varphi(x) = \frac{a_1}{x} + \dots + \frac{a_n}{x^n} \quad .$$

Applying (5.6) to our function $f(x) = \varphi(x) + O(x^{-n-1})$ and using (5.2), we obtain that

$$f^{(k)}(x) = \Delta(x;\varphi) + O(x^{-n-1}) \quad . \tag{5.7}$$

However, if we apply (5.6) to the function $\varphi(x)$ we obtain

$$\varphi^{(k)}(x) = \Delta(x;\varphi) + O(x^{-n-1}), \quad \text{or} \quad \Delta(x;\varphi) = \varphi^{(k)}(x) + O(x^{-n-1}) \quad .$$

Substituting this into (5.7) we finally have

$$f^{(k)}(x) = \varphi^{(k)}(x) + O(x^{-n-1}) \quad ,$$

and this reduces to (C_k) if we incorporate with the remainder those terms of $\varphi^{(k)}(x)$ which are $O(x^{-n-1})$. This establishes (C_k).

By way of a counter-example we consider the function

$$f(x) = \sum_{1}^{n} a_\nu x^{-\nu} + (\sin e^x)/x^{n+1} \quad .$$

Observe that it satisfies all assumptions of Theorem 3 with the exception of (5.2). It is likewise evident by differentiations that none of the estimates (C_k) hold.

6. The differentiation of asymptotic expansions. Let $f(x)$ $(x > 0)$ admit the asymptotic expansion

$$(6.1) \qquad f(x) \sim \frac{a_1}{x} + \ldots + \frac{a_n}{x^n} + \ldots \quad \text{as} \quad \kappa \to \infty \quad .$$

This means that the estimates

$$(6.2) \qquad f(x) = \frac{a_1}{x} + \ldots + \frac{a_n}{x^n} + O(x^{-n-1})$$

hold for all values of $n = 0,1,\ldots$. It is even sufficient for (6.2) to hold for arbitrarily large values of n.

Let us assume that $f(x) \in C^\infty(0,\infty)$ and find conditions which will insure that the expansion (6.1) may be differentiated any number of times leading to the expansion

$$(6.3) \qquad f^{(k)}(x) \sim (-1)^k \frac{(1)_k a_1}{x^{k+1}} + \ldots + (-1)^k \frac{(n-k)_k a_{n-k}}{x^n} + \ldots \quad (k \geq 1) \quad .$$

A necessary condition for this to hold is that we have

$$(6.4) \qquad f^{(n)}(x) = O(x^{-n-1}) \quad \text{for} \quad n = 1,2,\ldots \quad .$$

As an application of Theorem 3 we will show not only that the conditions (6.4) are also sufficient, but that it is already sufficient that (6.4) should hold for some arbitrarily large values of n. This we state as

Theorem 4. *Let* $f(x) \in C^{\infty}(0,\infty)$ *admit the expansion* (6.1). *Then* (6.1) *may be differentiated leading to* (6.3) *for all* k, *provided that the estimate*

$$f^{(n)}(x) = 0(x^{-n-1}) \quad \text{if} \quad n \in N \,, \tag{6.5}$$

holds for some infinite set N *of integers.*

Proof: We assume (6.1), and (6.5) for $n \in N$. However, (6.1) implies that (6.2) holds. Let $k \leq n-1$. By Theorem 3 (6.2) and (6.5) imply that we can differentiate (6.2) leading to

$$f^{(k)}(x) = (-1)^k \frac{(1)_k a_1}{x^{k+1}} + \ldots + (-1)^k \frac{(n-k)_k a_{n-k}}{x^n} + 0(x^{-n-1}) \quad . \tag{6.6}$$

Since $n \in N$ may be chosen arbitrarily large, this establishes the asymptotic expansion (6.3), proving the theorem.

For other conditions insuring the differentiability of asymptotic expansion see [3, theorem 3 on p. 542].

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report presents two further applications of very elementary formulae of approximate differentiation. The first is a new derivation in a somewhat sharper form of the following theorem of V. M. Olovyanišnikov:

Let $N_n (n \geq 2)$ be the class of functions $g(x)$ such that $g(x)$, $g'(x),\ldots,g^{(n)}(x)$ are ≥ 0, bounded, and non-decreasing on the half-line $-\infty < x \leq 0$. A special element of N_n is

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ABSTRACT (continued)

$$g_*(x) = 0 \quad \text{if} \quad -\infty < x < -1, \; g_*(x) = (1+x)^n \quad \text{if} \quad -1 \leqq x \leqq 0 \quad .$$

If $g(x) \in N_n$ is such that

$$g(0) < g_*(0) = 1, \; g^{(n)}(0) < g_*^{(n)}(0) = n!$$

then

(1) $$g^{(\nu)}(0) \leqq g_*^{(\nu)}(0) \quad \text{for} \quad \nu = 1,\ldots,n-1 \quad .$$

Moreover, if we have equality in (1) for some value of ν, then we have there equality for all ν, and this happens only if $g(x) = g_*(x)$ in $(-\infty,0]$.

The second application gives sufficient conditions for the differentiability of asymptotic expansion (Theorem 4).

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